Are NOON-States Entangled?

Overview

- **Entanglement for Bipartite Systems**
- **Examples**
- **A Bell Inequality for Homodyne Tomography**
- **Do NOON-States Violate a Bell Inequality?**

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Entanglement

Following Werner and Wolf: A state is called **separable** or *classically correlated*, if it can be written as a convex combination of tensor product states $W = \sum_{r=1}^{n} p_r W_r^1 \otimes W_r^2$. Otherwise, it is simply called **entangled**.

A joint measurement of the observables A^1 and A^2 on the respective subsystems may be written as

$$\mathrm{tr}(W \cdot A^1 \otimes A^2) = \sum_{r=1}^{n} p_r \mathrm{tr}(W_r^1 A^1)\mathrm{tr}(W_r^2 A^2)$$

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Example for a nonseparable state:

$$|\Psi^{12}\rangle = \frac{1}{\sqrt{2}}(|\uparrow^1, \downarrow^2\rangle - |\downarrow^1, \uparrow^2\rangle) \stackrel{?}{=} \left(\alpha|\uparrow^1\rangle + \beta|\downarrow^1\rangle\right) \otimes \left(\gamma|\uparrow^2\rangle + \delta|\downarrow^2\rangle\right).$$

$$\frac{1}{\sqrt{2}}(|\uparrow^1, \downarrow^2\rangle - |\downarrow^1, \uparrow^2\rangle) \stackrel{?}{=} \alpha\gamma|\uparrow^1, \uparrow^2\rangle + \beta\delta|\downarrow^1, \downarrow^2\rangle + \alpha\delta|\uparrow^1, \downarrow^2\rangle + \beta\gamma|\downarrow^1, \uparrow^2\rangle.$$

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$$\frac{1}{\sqrt{2}}(|\uparrow^1, \downarrow^2\rangle - |\downarrow^1, \uparrow^2\rangle) \neq \left(\alpha|\uparrow^1\rangle + \beta|\downarrow^1\rangle\right) \otimes \left(\gamma|\uparrow^2\rangle + \delta|\downarrow^2\rangle\right).$$

Separability and Bell inequalities

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This conjecture is false!

Conclusion: There are different kinds of entanglement. Entangled states that do not violate a Bell inequality and those that violate a Bell inequality.

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CHSH form of Bell inequalities with operators A_i, A'_i on H_i $(i = 1, 2)$ and $-1 \leq A_i, A'_i \leq 1$.

$$\mathrm{tr}\rho \left(A_1 \otimes A_2 + A'_1 \otimes A_2 + A_1 \otimes A'_2 - A'_1 \otimes A'_2 \right) \leq 2$$

$$\langle A_1 \otimes A_2 \rangle + \langle A'_1 \otimes A_2 \rangle + \langle A_1 \otimes A'_2 \rangle - \langle A'_2 \otimes A'_1 \rangle \leq 2$$

A classical correlation experiment

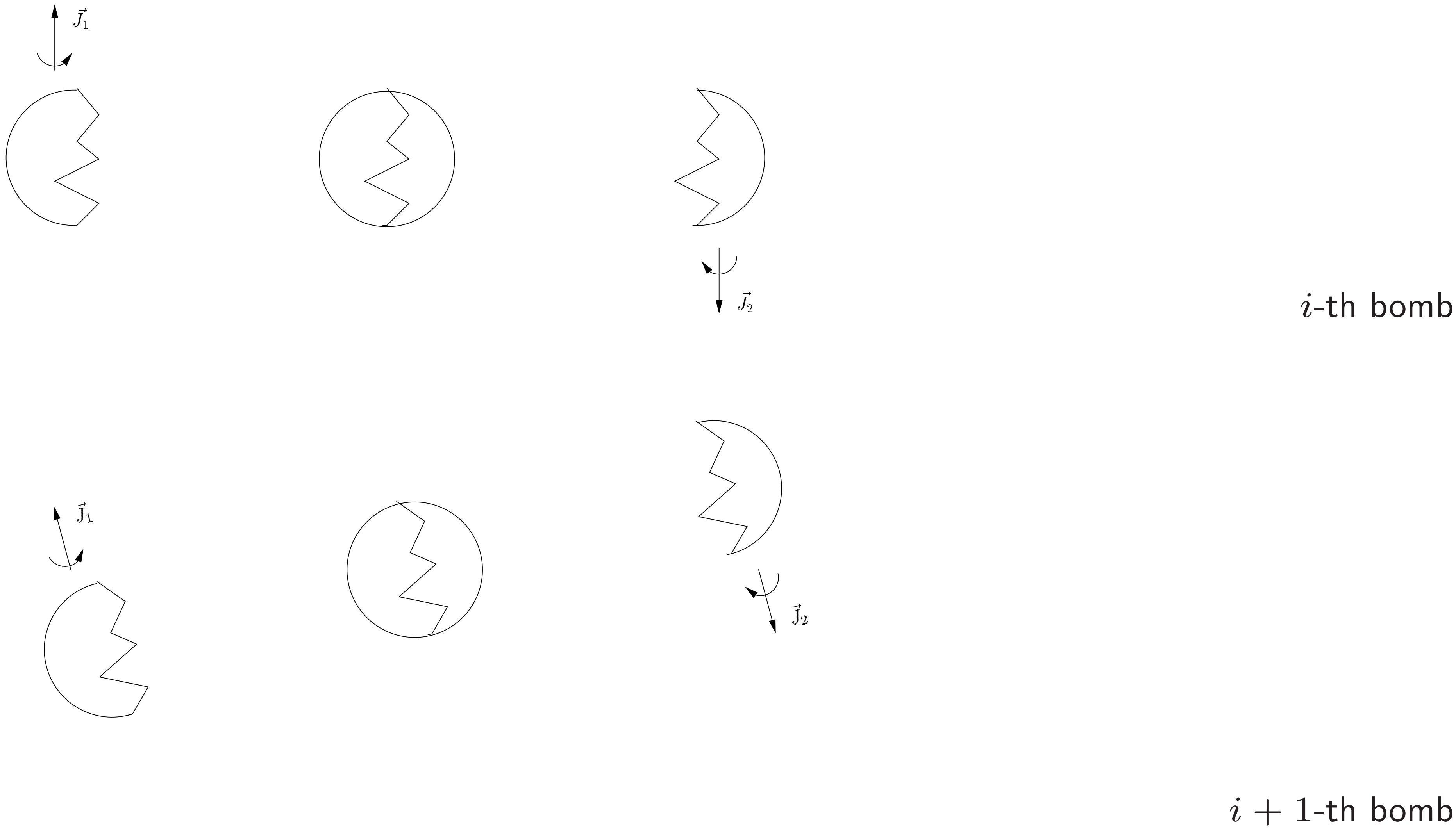

Figure 1: We assume that the bomb does not carry angular momentum before it explodes, i.e. $\vec{J}_1 + \vec{J}_2 = 0$.

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A classical correlation experiment

Figure 2: Measurement of the correlated angular momentum

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A classical correlation experiment

Figure 3: Measurement of the correlated angular momentum

Calculation of the correlation function:

$$\mathrm{cov}(\vec{a}\cdot\vec{J}_1, \vec{b}\cdot\vec{J}_2) = \Big\langle (\vec{a}\cdot\vec{J}_1 - \langle\vec{a}\cdot\vec{J}_1\rangle)(\vec{b}\cdot\vec{J}_2 - \langle\vec{b}\cdot\vec{J}_2\rangle) \Big\rangle \ ,$$

under the condition $\vec{J}_1 = -\vec{J}_2$.

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Figure 4: Measurement of the correlated angular momentum

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The mean values vanish, i.e. $\langle\vec{a}\cdot\vec{J}_1\rangle = 0$, $\langle\vec{b}\cdot\vec{J}_2\rangle = 0$, reducing the expression to

$$\text{cov}(\vec{a}\cdot\vec{J}_1, \vec{b}\cdot\vec{J}_2) = \langle\vec{a}\cdot\vec{J}_1\vec{b}\cdot\vec{J}_2\rangle = -\langle\vec{a}\cdot\vec{J}_1\vec{b}\cdot\vec{J}_1\rangle$$

A classical correlation experiment

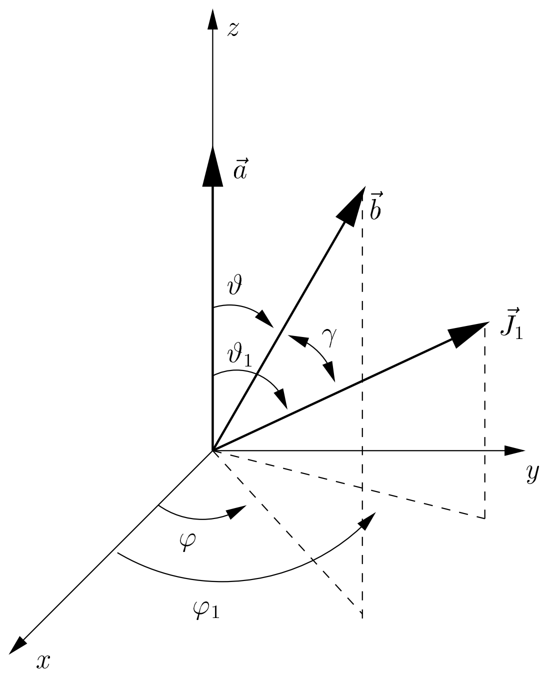

Figure 5: Representation of $\vec{a}$, $\vec{b}$ and $\vec{J}_1$ in spherical coordinates

$\vec{a} \cdot \vec{J}_1 = \cos(\vartheta_1)$, $\vec{b} \cdot \vec{J}_1 = \cos(\gamma)$, for $\vec{a}, \vec{b}, \vec{J}_1$ and $\vec{J}_2$ *unit vectors*,
$\cos(\gamma) = \cos(\vartheta_1)\cos(\vartheta) + \sin(\vartheta_1)\sin(\vartheta)\cos(\varphi_1 - \varphi)$.

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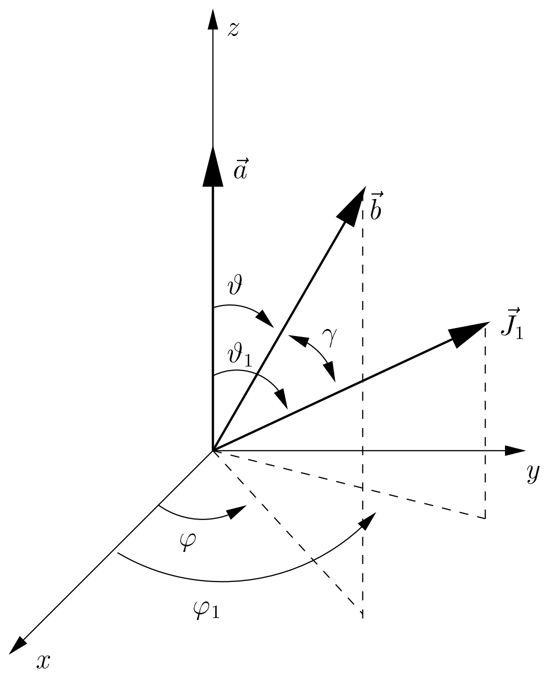

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$$\mathrm{cov}(\vec{a} \cdot \vec{J}_1, \vec{b} \cdot \vec{J}_2) = -\langle \vec{a} \cdot \vec{J}_1 \vec{b} \cdot \vec{J}_1 \rangle = -\frac{1}{4\pi} \iint_S \cos(\vartheta_1)\cos(\gamma) \mathrm{d}\cos(\vartheta_1) d\varphi_1 = -\frac{1}{3}\cos(\vartheta)\,.$$

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A quantum mechanical correlation experiment

Consider the wavefunction of two spin-$\frac{1}{2}$ particles in the spin singulett state of total spin 0:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle)\,.$$

Analogous to the classical observables $\vec{a}\cdot\vec{J}_1$ and $\vec{b}\cdot\vec{J}_2$ we consider the quantum mechanical observables $\vec{a}\cdot\vec{\sigma}_1$ and $\vec{b}\cdot\vec{\sigma}_2$.

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Bell inequality:

$$\begin{aligned}\langle\vec{a}\cdot\vec{\sigma}_1\vec{b}\cdot\vec{\sigma}_2\rangle + \langle\vec{a}\cdot\vec{\sigma}_1\vec{b}'\cdot\vec{\sigma}_2\rangle + \langle\vec{a}'\cdot\vec{\sigma}_1\vec{b}\cdot\vec{\sigma}_2\rangle - \langle\vec{a}'\cdot\vec{\sigma}_1\vec{b}'\cdot\vec{\sigma}_2\rangle &\leq 2\\ -\cos(\vartheta) - \cos(\vartheta+\vartheta_1) - \cos(\vartheta+\vartheta_2) + \cos(\vartheta+\vartheta_1+\vartheta_2) &\leq 2\end{aligned}$$

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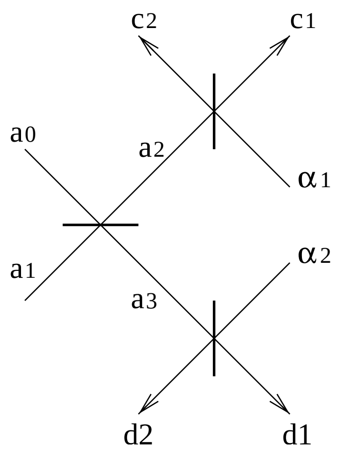
Maximally violated e.g. for $\vartheta = \frac{5}{4}\pi$, $\vartheta_1 = -\frac{\pi}{2}$, $\vartheta_2 = -\frac{\pi}{2}$.

$$-\cos(\frac{5}{4}\pi) - 2\cos(\frac{3\pi}{4}) + \cos(\frac{\pi}{4}) = 2\sqrt{2} \approx 2.83$$

Homodyne Tomography



Where $\alpha_1 = |\alpha| \mathrm{e}^{\mathrm{i}\vartheta}$ and $\alpha_2 = |\alpha| \mathrm{e}^{\mathrm{i}\varphi}$.

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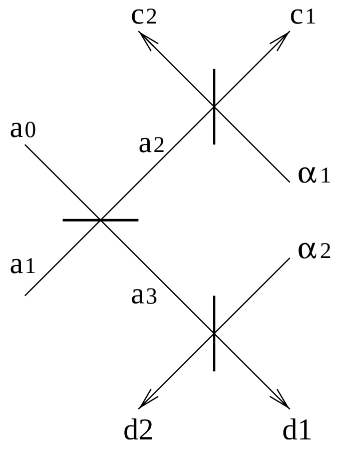

Where $\alpha_1 = |\alpha| \mathrm{e}^{\mathrm{i}\vartheta}$ and $\alpha_2 = |\alpha| \mathrm{e}^{\mathrm{i}\varphi}$. Reid and Walls derived a Bell inequality for the classical correlation function:

$$E(\vartheta, \varphi) = \frac{\langle (I_{c_1} - I_{c_2})(I_{d_1} - I_{d_2}) \rangle}{\langle (I_{c_1} + I_{c_2})(I_{d_1} + I_{d_2}) \rangle}$$

$$-2 \leq E(\vartheta, \varphi) - E(\vartheta, \varphi') + E(\vartheta', \varphi) + E(\vartheta', \varphi') \leq 2$$

Quantum mechanically, the expectation value can be written as

$$E(\vartheta, \varphi) = \frac{\langle : (c_1^\dagger c_1 - c_2^\dagger c_2)(d_1^\dagger d_1 - d_2^\dagger d_2) : \rangle}{\langle : (c_1^\dagger c_1 + c_2^\dagger c_2)(d_1^\dagger d_1 + d_2^\dagger d_2) : \rangle}$$

Homodyne Tomography

After some algebra:

$$E(\vartheta,\varphi) = -|\alpha|^2 \frac{\left\langle (\mathrm{e}^{\mathrm{i}\vartheta} a_2^\dagger - \mathrm{e}^{-\mathrm{i}\vartheta} a_2)(\mathrm{e}^{\mathrm{i}\varphi} a_3^\dagger - \mathrm{e}^{-\mathrm{i}\varphi} a_3) \right\rangle}{\left\langle (a_2^\dagger a_2 + |\alpha|^2)(a_3^\dagger a_3 + |\alpha|^2) \right\rangle}$$

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For the state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle + i|1,0\rangle)$, the expectation value turns out to be

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We may violate this inequality for $|\alpha|^2 < \sqrt{2} - 1$.

For an experiment see A.I. Lvovsky et al. PRL **92**, 193601 (2004).

Homodyne Tomography for $N > 1$

For $|\Psi^N\rangle = \frac{1}{\sqrt{2}}(|N, 0\rangle + e^{i\varphi}|0, N\rangle)$

$$E(\vartheta, \varphi) = -|\alpha|^2 \frac{\left\langle (e^{i\vartheta} a_2^\dagger - e^{-i\vartheta} a_2)(e^{i\varphi} a_3^\dagger - e^{-i\varphi} a_3) \right\rangle}{\left\langle (a_2^\dagger a_2 + |\alpha|^2)(a_3^\dagger a_3 + |\alpha|^2) \right\rangle} = 0$$

vanishes, unfortunately!, i.e. the Bell inequality is not violated like in the N=1 case!

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Conclusion: With this kind of tomography we may not show that NOON-states violate the respective Bell inequality for $N > 1$.

But that does not mean that NOON-states do not violate a Bell inequality in general of course! We only need to look for other observables and derive a Bell inequality for them!

Useful Entanglement

$$\frac{1}{\sqrt{2}}(a_1^\dagger + a_2^\dagger)|0,0\rangle = \frac{1}{\sqrt{2}}(|1\rangle_\uparrow|0\rangle_\rightarrow + |0\rangle_\uparrow|1\rangle_\rightarrow) \quad .$$

$$A_1^\dagger = \frac{1}{\sqrt{2}}(a_1^\dagger + a_2^\dagger) \quad ,$$

$$A_2^\dagger = \frac{1}{\sqrt{2}}(-a_1^\dagger + a_2^\dagger) \quad .$$

$A_1^\dagger$ und $A_2^\dagger$ are related to $a_1^\dagger$ and $a_2^\dagger$ by a simple rotation of 45° $|\,\rangle_\nearrow|\,\rangle_\nwarrow$ and we obtain

$$|1\rangle_\nearrow|0\rangle_\nwarrow = \frac{1}{\sqrt{2}}(|1\rangle_\uparrow|0\rangle_\rightarrow + |0\rangle_\uparrow|1\rangle_\rightarrow) \quad .$$

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Conclusion: In order to have a useful entangled state we need to have spatially separated modes.

End!

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