Generation of N-Photon Entangled States in a Two-Mode Jaynes–Cummings Model

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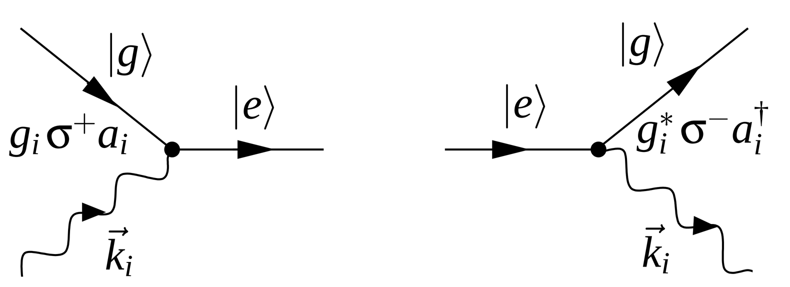
1. Algebraic Solution of the Two-Mode Jaynes–Cummings Model

The JC Hamiltonian for resonant interaction of a two-level atom ($|e\rangle, |g\rangle$) with two field modes (a_1, a_2), is given by $H = H_0 + H_{\text{int}}$, where

$$H_0 = \hbar\omega\left(\frac{\sigma_z + \mathbf{1}}{2} + (a_1^\dagger a_1 + a_2^\dagger a_2)\mathbf{1}\right),$$
$$H_{\text{int}} = \hbar\left(\sigma^+(g_1 a_1 + g_2 a_2) + \sigma^-\left(g_1^* a_1^\dagger + g_2^* a_2^\dagger\right)\right).$$



Feynman representations of the interaction operator H_{int}.

The operators for the atom are defined by $\sigma_z := |e\rangle\langle e| - |g\rangle\langle g|$, $\sigma^+ := |e\rangle\langle g|$ and $\sigma^- := |g\rangle\langle e|$. We introduce quasi-mode operators:

$$A_1 = \gamma_1 a_1 + \gamma_2 a_2, \quad A_2 = -\gamma_2^* a_1 + \gamma_1^* a_2, \qquad (1)$$

where $\gamma_i := g_i/g$, $g := \sqrt{|g_1|^2 + |g_2|^2}$. This is an SU(2) transformation of the mode operators a_1, a_2. The transformed Hamiltonian reads

$$H_0 = \hbar\omega\left(\frac{\sigma_z + \mathbf{1}}{2} + (A_1^\dagger A_1 + A_2^\dagger A_2)\mathbf{1}\right),$$
$$H_{\text{int}} = \hbar g\left(\sigma^+ A_1 + \sigma^- A_1^\dagger\right),$$

representing a JC Hamiltonian for quasi-mode one decoupled from a non-interacting quasi-mode two. The time-evolution operator in the interaction picture is then the same as for a one quasi-mode JC model:

$$U = \exp(-\mathrm{i}H_{\text{int}}t/\hbar) =$$
$$\cos\left(\tau\sqrt{A_1^\dagger A_1 + 1}\right)|e\rangle\langle e| + \frac{\sin\left(\tau\sqrt{A_1^\dagger A_1 + 1}\right)}{\mathrm{i}\sqrt{A_1^\dagger A_1 + 1}} A_1 |e\rangle\langle g|$$
$$+ A_1^\dagger \frac{\sin\left(\tau\sqrt{A_1^\dagger A_1 + 1}\right)}{\mathrm{i}\sqrt{A_1^\dagger A_1 + 1}} |g\rangle\langle e| + \cos\left(\tau\sqrt{A_1^\dagger A_1}\right)|g\rangle\langle g|.$$

Definition of the quasi-mode Fock states:

$$|n_1, n_2\rangle\!\rangle := \frac{A_1^{\dagger n_1} A_2^{\dagger n_2}}{\sqrt{n_1! n_2!}} |0,0\rangle\!\rangle.$$

To find the transformation between the two-mode Fock states $|n_1, n_2\rangle$ and the two-quasi-mode Fock states $|n_1, n_2\rangle\!\rangle$, we use Schwinger's oscillator model and introduce angular momentum states $|j,m\rangle$ and $|j,m\rangle\!\rangle$, where $j = (n_1 + n_2)/2$ and $m = (n_1 - n_2)/2$. We obtain the important relation between the quasi-mode and the mode Fock bases.

$$|j,m\rangle\!\rangle = \sum_{m'=-j}^{j} D_{m'm}^{(j)}(\varphi, \vartheta, \chi)\,|j,m'\rangle, \qquad (2)$$
$$|j,m\rangle = \sum_{m'=-j}^{j} D_{m'm}^{(j)\dagger}(\varphi, \vartheta, \chi)\,|j,m'\rangle\!\rangle. \qquad (3)$$

Here $D_{m'm}^{(j)}(\varphi, \vartheta, \xi) = \exp[-\mathrm{i}(m'\varphi + m\chi)]\,d_{m'm}^{(j)}(\vartheta)$ are the Wigner D-matrix elements of the SU(2) group with arguments determined by $\varphi = \varphi_1 - \varphi_2$, $\chi = \varphi_1 + \varphi_2$, $\cos(\vartheta/2) := |\gamma_1|$, $\sin(\vartheta/2) := |\gamma_2|$, and $\gamma_i = |\gamma_i|\exp(\mathrm{i}\varphi_i)$.

The action of U_{ab} on the field states is easily calculated in the quasi-mode Fock basis

$$U_{ee}(\tau)\,|j,m\rangle\!\rangle = \cos\left(\tau\sqrt{j+m+1}\right)|j,m\rangle\!\rangle,$$
$$U_{ge}(\tau)\,|j,m\rangle\!\rangle = -\mathrm{i}\sin\left(\tau\sqrt{j+m+1}\right)|j+\tfrac{1}{2}, m+\tfrac{1}{2}\rangle\!\rangle,$$
$$U_{eg}(\tau)\,|j,m\rangle\!\rangle = -\mathrm{i}\sin\left(\tau\sqrt{j+m}\right)|j-\tfrac{1}{2}, m-\tfrac{1}{2}\rangle\!\rangle,$$
$$U_{gg}(\tau)\,|j,m\rangle\!\rangle = \cos\left(\tau\sqrt{j+m}\right)|j,m\rangle\!\rangle, \qquad (4)$$

showing that U_{ee} and U_{gg} do not change the number of quasi-photons, whereas U_{ge} (U_{eg}) act as creation (annihilation) operators of quasi-mode one. We find for the action on the usual Fock states

$$U_{ee}(\tau)\,|j,m\rangle = \sum_{m'=-j}^{j} C_{m'm}^{j}(\tau)\,|j,m'\rangle,$$
$$U_{ge}(\tau)\,|j,m\rangle = \sum_{m'=-j-\frac{1}{2}}^{j+\frac{1}{2}} S_{m'm}^{j}(\tau)\,|j+\tfrac{1}{2}, m'\rangle,$$
$$U_{eg}(\tau)\,|j,m\rangle = \sum_{m'=-j+\frac{1}{2}}^{j-\frac{1}{2}} \overline{S}_{m'm}^{j}(\tau)\,|j-\tfrac{1}{2}, m'\rangle,$$
$$U_{gg}(\tau)\,|j,m\rangle = \sum_{m'=-j}^{j} \overline{C}_{m'm}^{j}(\tau)\,|j,m'\rangle, \qquad (5)$$

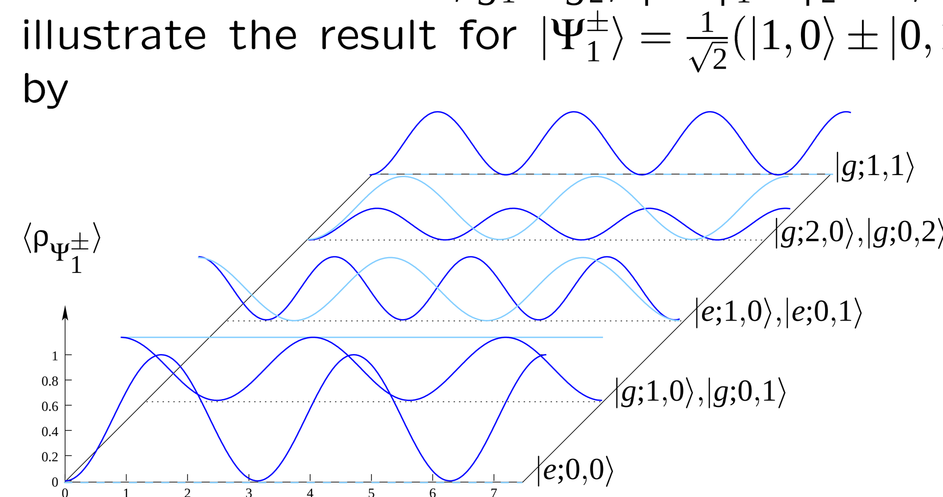
where we have introduced the following coefficients

$$C_{m'm}^{j}(\tau) = \sum_{\nu=-j}^{j} \cos\left(\tau\sqrt{j+\nu+1}\right) D_{m'\nu}^{(j)} D_{\nu m}^{(j)\dagger},$$
$$S_{m'm}^{j}(\tau) = -\mathrm{i}\sum_{\nu=-j}^{j} \sin\left(\tau\sqrt{j+\nu+1}\right) D_{m',\nu+\frac{1}{2}}^{(j+\frac{1}{2})} D_{\nu m}^{(j)\dagger},$$
$$\overline{S}_{m'm}^{j}(\tau) = -\mathrm{i}\sum_{\nu=-j}^{j} \sin\left(\tau\sqrt{j+\nu}\right) D_{m',\nu-\frac{1}{2}}^{(j-\frac{1}{2})} D_{\nu m}^{(j)\dagger},$$
$$\overline{C}_{m'm}^{j}(\tau) = \sum_{\nu=-j}^{j} \cos\left(\tau\sqrt{j+\nu}\right) D_{m'\nu}^{(j)} D_{\nu m}^{(j)\dagger}. \qquad (6)$$

With the help of this compact formalism, all expectation values can be calculated analytically.

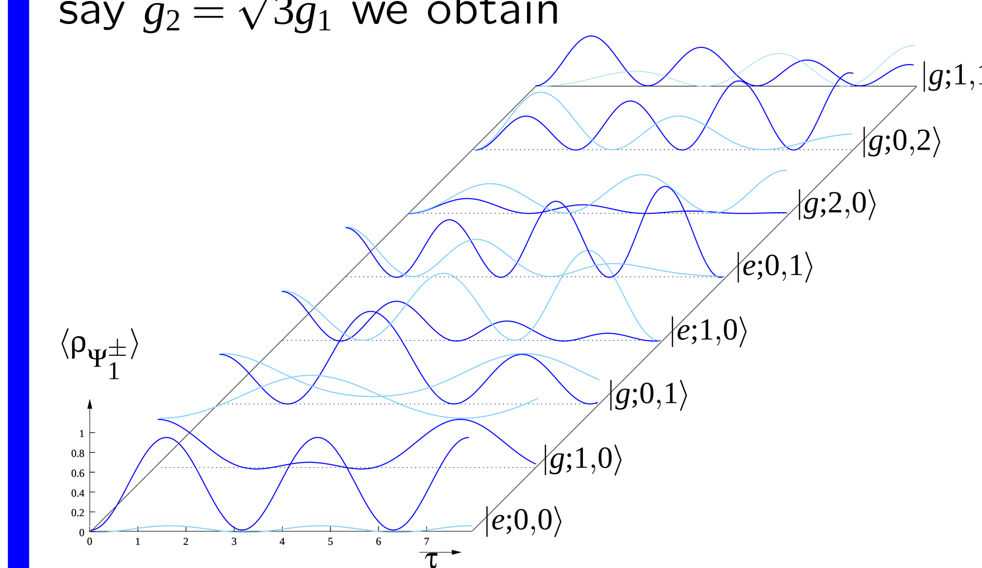
2. Generation of the NOON-States $|\Psi_N^{\pm}\rangle = \frac{1}{\sqrt{2}}(|N,0\rangle \pm |0,N\rangle)$ in a Single Step

We show that it is possible to entangle N photons in two field modes in a single shot of an atom through a two-mode cavity. This property of the two-mode JC model can be understood if we think of the atom (re)emitting photons into and (re)absorbing photons from the two modes many times during the interaction time τ.
For the case of $N = 1$, $g_1 = g_2$, $\varphi = \varphi_1 - \varphi_2 = 0$, we illustrate the result for $|\Psi_1^{\pm}\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle \pm |0,1\rangle)$ by



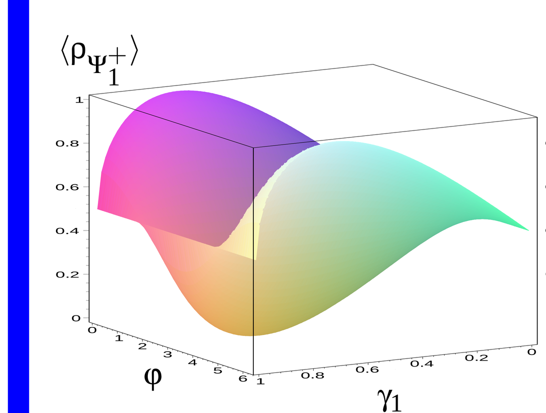
Parametric plot of the generation probabilities $\langle\rho_{\Psi_1^+}\rangle$ (dark-blue) and $\langle\rho_{\Psi_1^-}\rangle$ (light-blue) as function of time $\tau = gt$, for different initial atom-field states shown on the right.

If the two coupling constants are different, say $g_2 = \sqrt{3}g_1$ we obtain

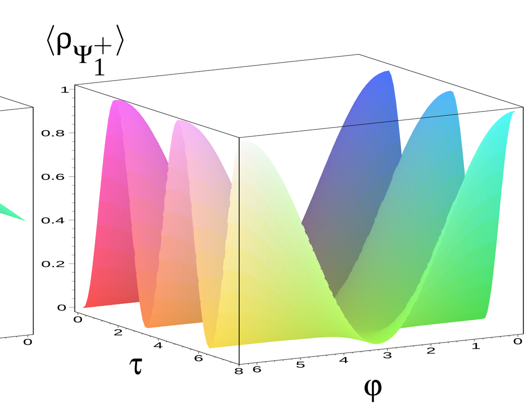


Parametric plot of the generation probabilities $\langle\rho_{\Psi_1^+}\rangle$ (dark-blue) and $\langle\rho_{\Psi_1^-}\rangle$ (light-blue) as function of time $\tau = gt$, for different initial atom-field states shown on the right.

The dependence of the generation probability on the relative phase and coupling strength, or the interaction time and relative phase can also be investigated.

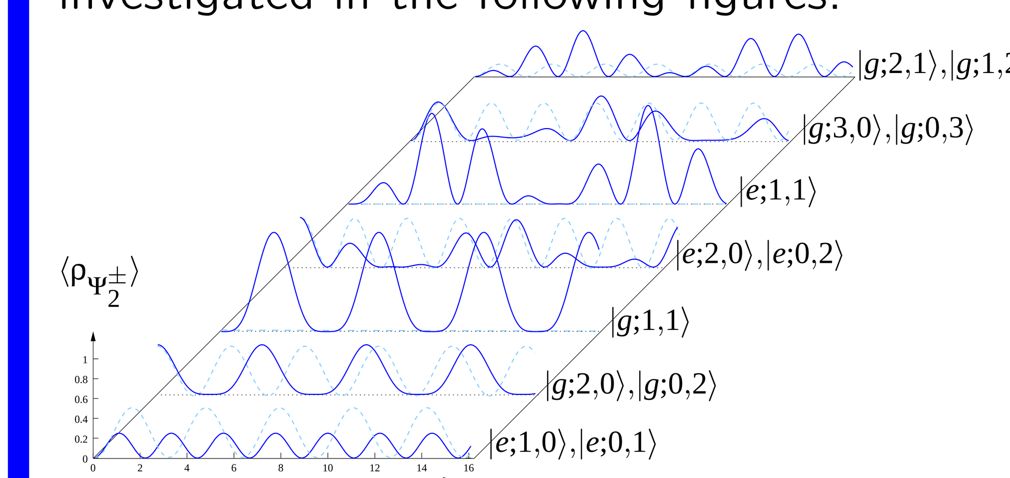


Generation probability for the state $|\Psi_1^+\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle + |0,1\rangle)$ taking initially $|e;0,0\rangle$ and a fixed interaction time $\tau = \pi/2$, as a function of the relative phase $\varphi = \varphi_1 - \varphi_2$ and γ_1.

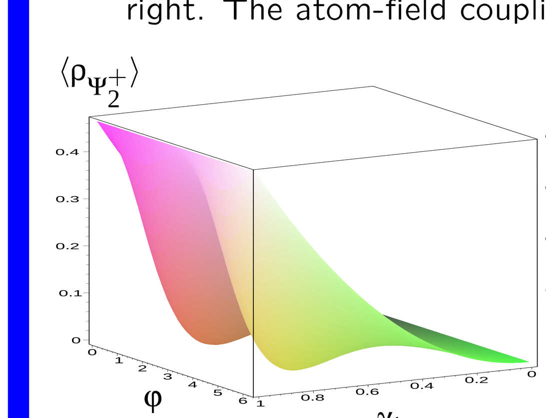


Generation probability for the state $|\Psi_1^+\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle + |0,1\rangle)$ taking initially $|e;0,0\rangle$ and a fixed coupling constant $\gamma_1 = 1/\sqrt{2}$ as a function of the interaction time τ and relative phase $\varphi = \varphi_1 - \varphi_2$.

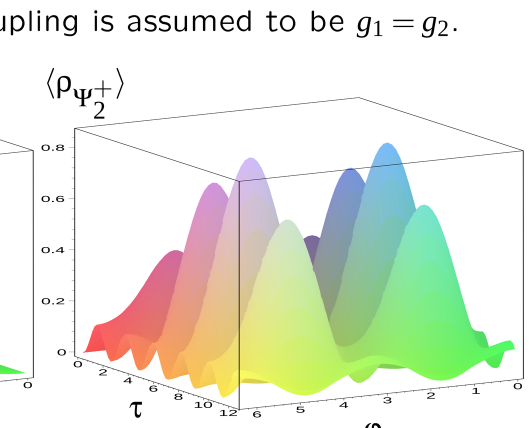
The generation of NOON-states with larger N, e.g., $N = 2$, i.e., $|\Psi_2^{\pm}\rangle = \frac{1}{\sqrt{2}}(|2,0\rangle \pm |0,2\rangle)$, is investigated in the following figures.



Parametric plot of the generation probabilities $\langle\rho_{\Psi_2^+}\rangle$ (dark-blue) and $\langle\rho_{\Psi_2^-}\rangle$ (light-blue) as function of time $\tau = gt$, for different initial atom-field states shown on the right. The atom-field coupling is assumed to be $g_1 = g_2$.



Generation probability for the state $|\Psi_2^+\rangle = \frac{1}{\sqrt{2}}(|2,0\rangle + |0,2\rangle)$ taking initially $|e;1,0\rangle$ and a fixed interaction time $\tau = \pi$, as a function of the relative phase $\varphi = \varphi_1 - \varphi_2$ and γ_1.



Generation probability for the state $|\Psi_2^+\rangle = \frac{1}{\sqrt{2}}(|2,0\rangle + |0,2\rangle)$ taking initially $|e;1,0\rangle$ and a fixed coupling constant $\gamma_1 = 1/2$ as a function of the interaction time τ and relative phase $\varphi = \varphi_1 - \varphi_2$.

In general, given an initial atom-field state one has to apply a maximization method to find the set of parameters (τ, φ, γ_1) where the generation probability has a maximum.
For general results and for conditional and non-conditional generation schemes see:

References

[1] C. Wildfeuer and D. H. Schiller, quant-ph/0210138, to be published in Phys. Rev. A, May 2003

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