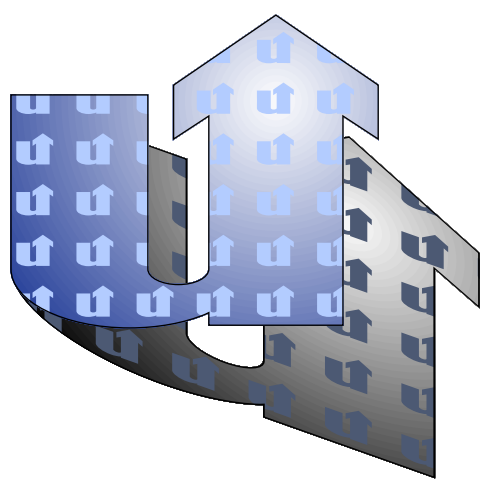




Generation of N -Photon Entangled States in a Two-Mode Jaynes–Cummings Model

C. Wildfeuer and D. H. Schiller
Fachbereich Physik, Universität Siegen, Germany

1. Entangled states

We consider the interaction of a two-level atom with two degenerate modes of a lossless cavity. We solve the corresponding Jaynes–Cummings (JC) model by an SU(2) transformation and discuss the generation of entangled states with a total number N of photons in the two modes:

$$|\Psi_N\rangle = \sum_{k=0}^N c_k^{(N)} |N-k, k\rangle, \quad (1)$$

which comprise also the maximally entangled Bell-states

$$|\Psi_N^\pm\rangle = \frac{1}{\sqrt{2}} (|N, 0\rangle \pm |0, N\rangle). \quad (2)$$

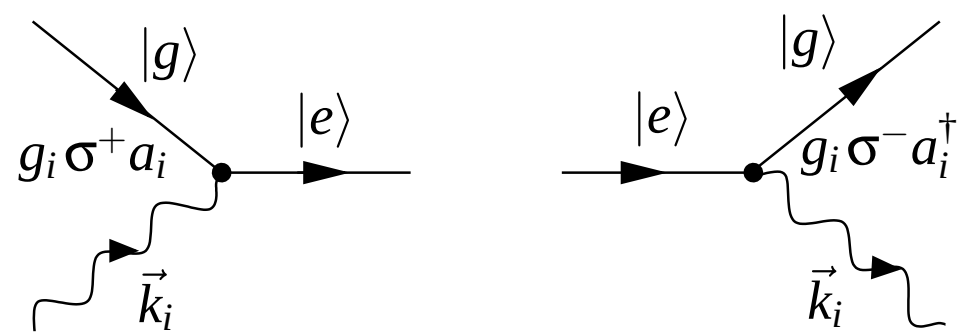
The states (1) and (2) are of great interest in the field of quantum optical lithography where they help to overcome the classical Rayleigh diffraction limit by a factor of $1/N$, N being the total number of photons in the two-mode field states.

2. Reduction to a one-mode interaction

The JC Hamiltonian for resonant interaction of a two-level atom ($|e\rangle, |g\rangle$) with two field modes (a_1, a_2), is given by $H = H_0 + H_{\text{int}}$, where

$$H_0 = \hbar\omega \left(\frac{\sigma_z + \mathbf{1}}{2} + (a_1^\dagger a_1 + a_2^\dagger a_2) \mathbf{1} \right),$$

$$H_{\text{int}} = \hbar \left(\sigma^+ (g_1 a_1 + g_2 a_2) + \sigma^- (g_1^* a_1^\dagger + g_2^* a_2^\dagger) \right).$$



Feynman representations of the interaction operator H_{int} .

The operators for the atom are defined by $\sigma_z := |e\rangle\langle e| - |g\rangle\langle g|$, $\sigma^+ := |e\rangle\langle g|$ and $\sigma^- := |g\rangle\langle e|$. We introduce quasi-mode operators:

$$A_1 = \gamma_1 a_1 + \gamma_2 a_2, \quad A_2 = -\gamma_2^* a_1 + \gamma_1^* a_2, \quad (3)$$

where $\gamma_i := g_i/g$, $g := \sqrt{|g_1|^2 + |g_2|^2}$. This is an SU(2) transformation of the mode operators a_1, a_2 . The transformed Hamiltonian then reads

$$H_0 = \hbar\omega \left(\frac{\sigma_z + \mathbf{1}}{2} + (A_1^\dagger A_1 + A_2^\dagger A_2) \mathbf{1} \right),$$

$$H_{\text{int}} = \hbar g \left(\sigma^+ A_1 + \sigma^- A_1^\dagger \right),$$

representing a JC Hamiltonian for quasi-mode one decoupled from a non-interacting quasi-mode two. The time-evolution operator in the interaction picture is then the same as for a one quasi-mode JC model:

$$U = \exp(-iH_{\text{int}}t/\hbar) =$$

$$\cos \left(\tau \sqrt{A_1^\dagger A_1 + 1} \right) |e\rangle\langle e| + \frac{\sin \left(\tau \sqrt{A_1^\dagger A_1 + 1} \right)}{i \sqrt{A_1^\dagger A_1 + 1}} A_1 |e\rangle\langle g|$$

$$+ A_1^\dagger \frac{\sin \left(\tau \sqrt{A_1^\dagger A_1 + 1} \right)}{i \sqrt{A_1^\dagger A_1 + 1}} |g\rangle\langle e| + \cos \left(\tau \sqrt{A_1^\dagger A_1} \right) |g\rangle\langle g|.$$

2.1 Properties of the quasi-mode operators

The quasi-mode operators $A_i, A_i^\dagger$, $i = 1, 2$, obey the same algebra as the mode operators $a_i, a_i^\dagger$, so that two-quasi-mode Fock states (denoted by a double-ket) can be defined by

$$|n_1, n_2\rangle\rangle := \frac{A_1^{\dagger n_1} A_2^{\dagger n_2}}{\sqrt{n_1! n_2!}} |0, 0\rangle\rangle.$$

To find the transformation between the two-mode Fock states $|n_1, n_2\rangle$ and the two-quasi-mode Fock states $|n_1, n_2\rangle\rangle$, we use Schwinger's oscillator model and introduce angular momentum states $|j, m\rangle$ and $|j, m\rangle\rangle$, where $j = (n_1 + n_2)/2$ and $m = (n_1 - n_2)/2$. We obtain the important relation between the quasi-mode and the mode Fock bases.

$$|j, m\rangle\rangle = \sum_{m'=-j}^j D_{m'm}^{(j)}(\varphi, \vartheta, \chi) |j, m'\rangle, \quad (5)$$

$$|j, m\rangle = \sum_{m'=-j}^j D_{m'm}^{(j)\dagger}(\varphi, \vartheta, \chi) |j, m'\rangle\rangle. \quad (6)$$

Here $D_{m'm}^{(j)}(\varphi, \vartheta, \chi) = \exp[-i(m'\varphi + m\chi)] d_{m'm}^{(j)}(\vartheta)$ are the Wigner D -matrix elements of the SU(2) group with arguments determined by $\varphi = \varphi_1 - \varphi_2$, $\chi = \varphi_1 + \varphi_2$, $\cos(\vartheta/2) := |\gamma_1|$, $\sin(\vartheta/2) := |\gamma_2|$, and $\gamma_i = |\gamma_i| \exp(i\varphi_i)$. The action of U_{ab} on the field states is easily calculated in the quasi-mode Fock basis

$$U_{ee}(\tau) |j, m\rangle\rangle = \cos \left(\tau \sqrt{j+m+1} \right) |j, m\rangle\rangle,$$

$$U_{ge}(\tau) |j, m\rangle\rangle = -i \sin \left(\tau \sqrt{j+m+1} \right) |j + \frac{1}{2}, m + \frac{1}{2}\rangle\rangle,$$

$$U_{eg}(\tau) |j, m\rangle\rangle = -i \sin \left(\tau \sqrt{j+m} \right) |j - \frac{1}{2}, m - \frac{1}{2}\rangle\rangle,$$

$$U_{gg}(\tau) |j, m\rangle\rangle = \cos \left(\tau \sqrt{j+m} \right) |j, m\rangle\rangle, \quad (7)$$

showing that U_{ee} and U_{gg} do not change the number of quasi-photons, whereas U_{ge} (U_{eg}) act as creation (annihilation) operators of quasi-mode one. We find for the action on the usual Fock states

$$U_{ee}(\tau) |j, m\rangle = \sum_{m'=-j}^j C_{m'm}^j(\tau) |j, m'\rangle,$$

$$U_{ge}(\tau) |j, m\rangle = \sum_{m'=-j-\frac{1}{2}}^{j+\frac{1}{2}} S_{m'm}^j(\tau) |j + \frac{1}{2}, m'\rangle,$$

$$U_{eg}(\tau) |j, m\rangle = \sum_{m'=-j+\frac{1}{2}}^{j-\frac{1}{2}} \bar{S}_{m'm}^j(\tau) |j - \frac{1}{2}, m'\rangle,$$

$$U_{gg}(\tau) |j, m\rangle = \sum_{m'=-j}^j \bar{C}_{m'm}^j(\tau) |j, m'\rangle, \quad (8)$$

where we have introduced the following coefficients

$$C_{m'm}^j(\tau) = \sum_{v=-j}^j \cos \left(\tau \sqrt{j+v+1} \right) D_{m'v}^{(j)} D_{vm}^{(j)\dagger},$$

$$S_{m'm}^j(\tau) = -i \sum_{v=-j}^j \sin \left(\tau \sqrt{j+v+1} \right) D_{m',v+\frac{1}{2}}^{(j+\frac{1}{2})} D_{vm}^{(j)\dagger},$$

$$\bar{S}_{m'm}^j(\tau) = -i \sum_{v=-j}^j \sin \left(\tau \sqrt{j+v} \right) D_{m',v-\frac{1}{2}}^{(j-\frac{1}{2})} D_{vm}^{(j)\dagger},$$

$$\bar{C}_{m'm}^j(\tau) = \sum_{v=-j}^j \cos \left(\tau \sqrt{j+v} \right) D_{m'v}^{(j)} D_{vm}^{(j)\dagger}. \quad (9)$$

3. Generation of $|\Psi_N\rangle$

Let the atom-field system initially be in the state $|e; \xi\rangle$ or $|g; \xi\rangle$ and consider the states (1). We denote their pure state density matrix by $\rho_{\Psi_N} = |\Psi_N\rangle\langle\Psi_N|$.

From the time-evolved initial states $U|e; \xi\rangle$, $U|g; \xi\rangle$ we obtain the reduced field-state density matrix $\rho_F(t)$ by tracing out the atomic degrees of freedom. The expectation value for the generation of $|\Psi_N\rangle$ then reads $\langle\rho_{\Psi_N}\rangle = \text{Tr}(\rho_F(t)\rho_{\Psi_N})$.

It follows that in order to obtain non-vanishing probabilities at time τ , the initial field state must contain at least one of the Fock states from the set

$$\{ |0, N\rangle, |1, N-1\rangle, \dots, |N, 0\rangle \}$$

$$\cup \{ |0, N-1\rangle, |1, N-2\rangle, \dots, |N-1, 0\rangle \}$$

if the atom is initially in the excited state, or from the set

$$\{ |0, N\rangle, |1, N-1\rangle, \dots, |N, 0\rangle \}$$

$$\cup \{ |0, N+1\rangle, |1, N\rangle, \dots, |N+1, 0\rangle \}$$

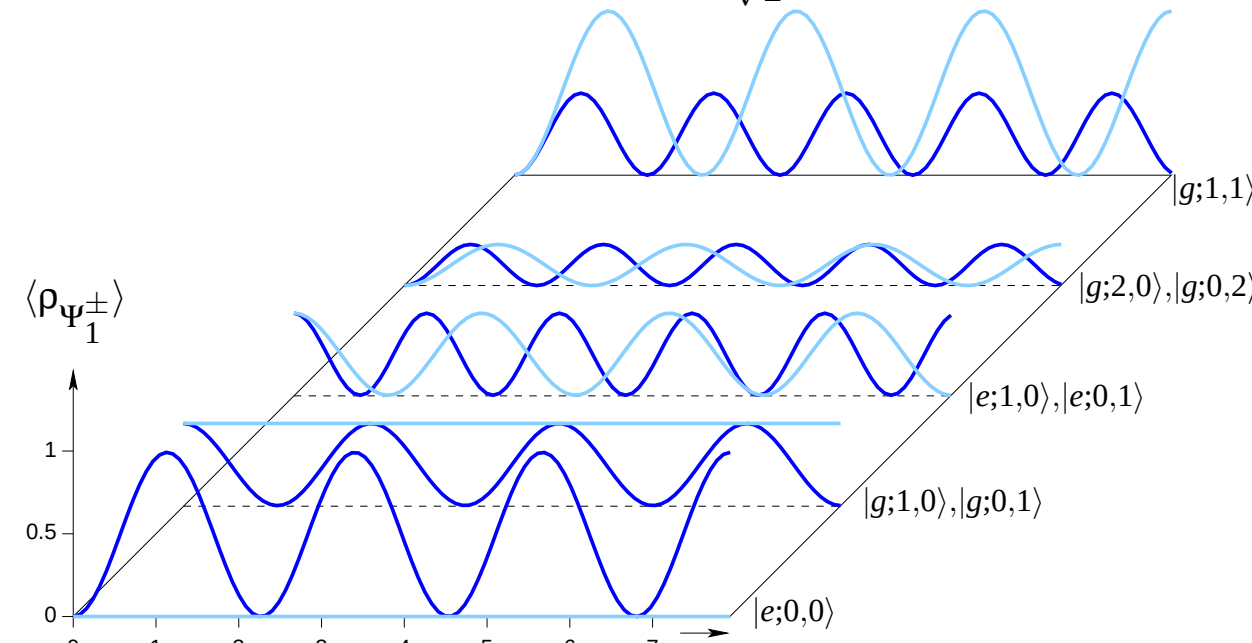
if it is in the ground state. As an example, we consider the creation of the Bell states in Eq. (2). Starting with the initial separable N -photon state $|e; N, 0\rangle = |e; \frac{N}{2}, \frac{N}{2}\rangle_S$, we obtain the entangled N -photon state $|\Psi_N^\pm\rangle$ with probability

$$\langle\rho_{\Psi_N^\pm}\rangle = \frac{1}{2} \left| C_{\frac{N}{2}, \frac{N}{2}}^{\frac{N}{2}}(\tau) \pm C_{-\frac{N}{2}, -\frac{N}{2}}^{\frac{N}{2}}(\tau) \right|^2, \quad (10)$$

as well as the entangled $(N+1)$ -photon state $|\Psi_{N+1}^\pm\rangle$ with probability

$$\langle\rho_{\Psi_{N+1}^\pm}\rangle = \frac{1}{2} \left| S_{\frac{N}{2}+1, \frac{N}{2}}^{\frac{N}{2}}(\tau) \pm S_{-\frac{N}{2}+1, \frac{N}{2}}^{\frac{N}{2}}(\tau) \right|^2. \quad (11)$$

In the case of Eq. (11) the Bell states $|\Psi_{N+1}^\pm\rangle$ have no overlap with the initial field state $|N, 0\rangle$. The probability at time τ , however, may come close to one for some particular values of the coupling constants and interaction time. In this case we may say that $|\Psi_{N+1}^\pm\rangle$ has been generated in a *single step* or single shot. For the case of $N=1$, $g_1=g_2$, $\varphi_1=\varphi_2=0$ (real coupling constants) we illustrate the result for $|\Psi_1^\pm\rangle = \frac{1}{\sqrt{2}}(|1, 0\rangle \pm |0, 1\rangle)$ by



The figure shows the probability $\langle\rho_{\Psi_1^\pm}\rangle$ for different initial atom-field states contributing to $N=1$. The dark-blue curves belong to the probability $\langle\rho_{\Psi_1^\pm}\rangle$ and the light-blue ones to $\langle\rho_{\Psi_1^\pm}\rangle$. On the x-axis we use the dimensionless time $\tau = g\tau$.

In an experimental realization the interaction time of the atom with the cavity-modes have to be adjusted to obtain a maximum. For general results and other schemes see:

References

- [1] C. Wildfeuer and D. H. Schiller,
quant-ph/0210138